The Mpemba effect in delayed cooling: The Descartes protocol

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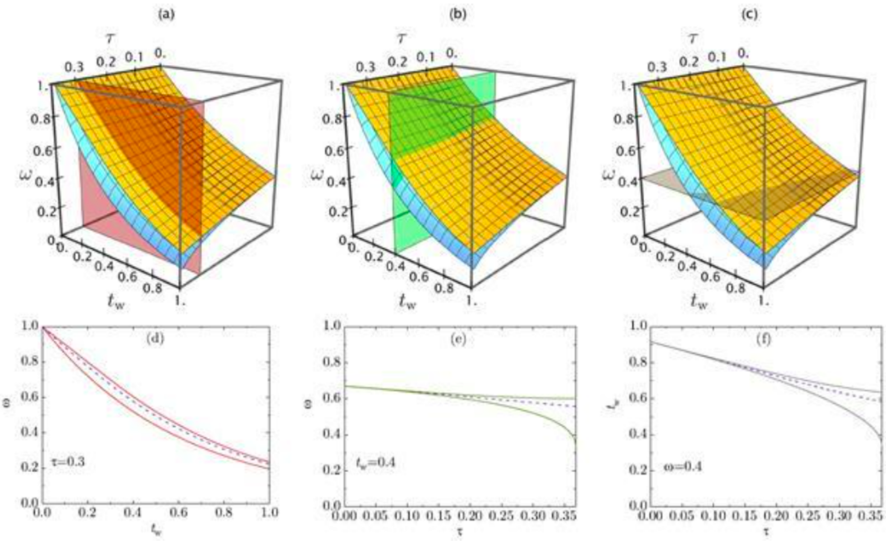
The Mpemba effect, whereby a hotter system cools faster than a colder one under identical conditions, continues to stimulate theoretical and experimental interest as a paradigmatic example of anomalous nonequilibrium relaxation. While most studies focus on systems brought into contact with a single thermal reservoir, recent works have shown that protocol design can strongly influence the occurrence and magnitude of the effect. In this talk I present a theoretical study of the Mpemba effect within the framework of the time-delayed Newton's law of cooling.

We introduce and analyze a three-reservoir scheme, termed the "Descartes protocol," in which two samples initially equilibrated at different temperatures are subjected to single-step quenches at different times before relaxing toward a common cold reservoir [1]. This protocol provides a transparent way to disentangle the roles of three independent control parameters: the delay time associated with the thermal response, the waiting time between the preparation of the samples, and the intermediate (warm) reservoir temperature. Using the exact solution of the delayed cooling equation, we derive analytical conditions for the existence of both the direct and the inverse Mpemba effects under instantaneous quenches. These conditions can be expressed as bounds on the normalized warm temperature for given values of the delay and waiting times.

Within the admissible region where the Mpemba effect occurs, the magnitude of the effect can be optimized by a suitable choice of the protocol parameters. In particular, we determine the value of the warm temperature that maximizes the effect for fixed delay and waiting times, and we obtain compact and accurate approximations for both this optimal parameter and the corresponding maximal magnitude of the temperature difference between the samples. An especially simple result emerges for the global optimum: for a fixed delay time, the absolute maximum of the Mpemba effect is reached when the waiting time equals the delay time.

The Descartes protocol also allows a direct comparison with previously studied two-reservoir schemes. Despite possessing an additional control parameter, the three-reservoir protocol is shown to yield a smaller maximal magnitude of the effect than the corresponding two-reservoir setup with the same delay time. Finally, the analysis is extended to finite-rate quenches. In that case, strict equality of the bath conditions prevents a genuine Mpemba crossing, but an approximate effect persists when the time scale of the bath dynamics remains sufficiently short compared with the intrinsic delay time.

These results provide an analytically tractable framework for understanding how delayed thermal response and protocol design influence anomalous relaxation phenomena, and they suggest new ways to control and optimize Mpemba-type effects in simple thermal systems.



Reference: [1] A. Santos, The Mpemba effect in the Descartes protocol: a time-delayed Newton's law of cooling approach, J. Phys. A: Math. Theor. 59, 145201 (2026).