A Statistical-Mechanics Framework for the TSP via Connectivity Penalties

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The Travelling Salesman Problem (TSP) admits a natural reformulation as a variational matching problem of degree $\kappa = 2$ supplemented by the global requirement that the solution form a single Hamiltonian cycle [1]. Whereas the local degree constraint fits naturally into a maximum-entropy framework [2], the connectivity requirement is inherently non-local and cannot be encoded as a simple per-edge condition, which has so far limited the use of statistical-mechanics methods for this class of combinatorial problems.

Here we introduce a unified variational framework in which the cycle-connectivity requirement is enforced through an energetic penalty $\mu N_{\mathrm{conn}}(G)$ within a Boltzmann–Gibbs distribution over graphs, and we systematically compare three candidate penalties: a spectral term based on the log-determinant of the reduced graph Laplacian, a non-backtracking variant built from the Hashimoto matrix via the Ihara–Bass identity [3,4], and a directed-acyclic-graph (DAG) [5] trace penalty of the form $\sum_k \frac{\alpha^k \,\mathrm{Tr}(V^k)}{k}$.

Within an independent-edge mean-field ansatz, each penalty produces a closed self-consistent equation for the p_{ij}'s, in which the gradient of the connectivity term plays the role of an effective, p-dependent correction to the cost matrix; for the Laplacian penalty this correction coincides with the effective resistance [**?**] R^{eff}_{ij}, while for the DAG penalty it reduces to a power series in the mean-field adjacency matrix that admits an efficient Sinkhorn implementation [7]] in the directed (asymmetric TSP) case.

The Laplacian-based penalty offers a transparent spectral interpretation of cycle connectivity and provides a natural bridge between the statistical-mechanics formulation and classical results on random walks and effective resistance on graphs, making it a useful analytical reference. Building on this picture, we show that the DAG penalty is minimised exactly on Hamiltonian cycles and, combined with annealing in β and μ, yields tour lengths competitive with state-of-the-art heuristics (LKH-3, TGCA) on benchmark asymmetric instances.

These results indicate that connectivity-aware variational principles provide a principled and computationally viable route from statistical mechanics to hard combinatorial optimisation, and identify the trace-based DAG penalty as the most promising building block for further analytical and algorithmic developments.

References:

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