# Scaling, fractal dynamics, and critical exponents: A geometrical view of Phase Transition

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Moving beyond simple associations, researchers need tools to quantify how variables influence each other in space and time. Correlation functions provide a mathematical framework for characterizing these essential dependencies, revealing insights into causality, structure, and hidden patterns within complex systems. In physical systems with many degrees of freedom, such as gases, liquids, and solids, a statistical analysis of these correlations is essential. For a field $F(x,t)$ that depends on spatial position $x$ and time $t$, it is often necessary to understand the correlation with itself at another position and time $F(x_0, t_0)$. This specific function is called the autocorrelation function.

In this context, the autocorrelation function for order-parameter fluctuations, introduced by Fisher [1], provides an important mathematical framework for understanding the second-order phase transition at equilibrium. However, his analysis is restricted to a Euclidean space of dimension $d$, and an exponent $\eta$ is introduced to correct the spatial behavior of the correlation function at $T = T_c$.

In recent work, Lima et al. [2], demonstrated that at $T_c$ a fractal analysis is necessary for a complete description of the correlation function. In this study, we investigate the fundamental physics and mathematics underlying phase transitions, emphasizing the deep interplay between scaling behavior, critical exponents, and fractal geometry. In particular, we show that the application of modern fractional differentials allows us to write an equation for the correlation function that recovers the correct exponents below the upper critical dimension. We obtain the exact expression for the Fisher exponent $\eta$. Furthermore, we examine the Rushbrooke scaling relation, which has been questioned in certain magnetic systems, and, drawing on results from the Ising model, we confirm that both our relations and the Rushbrooke scaling law hold even when $d$ is not an integer [3].

References:

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