A grand-canonical solution to a class of random optimization problems

Enrico Maria Fenoaltea[1], Izat Baybusinov, Yi-Cheng Zhang

[1]University of Barcelona, Spain

We introduce a unified analytical framework for a broad class of random combinatorial optimization problems, including the matching problem [1], the Traveling Salesman Problem (TSP) [2], and the minimum-weight k-factor problem [3]. These problems, central in computer science and statistical physics, are typically computationally intractable due to complex global constraints and disorder in edge weights.

Our approach reformulates these optimization tasks as an arrangement model, where agents are assigned to nodes of a network and incur pairwise interaction costs depending on their adjacency. This representation enables a natural mapping to statistical mechanics, allowing the system to be studied through a grand-canonical ensemble. In this formulation, strict topological constraints (e.g., fixed node degrees) are relaxed via the introduction of node-dependent chemical potentials, which enforce constraints only on average.

We derive an analytically tractable partition function for the resulting system, avoiding the need for more intricate techniques such as the replica or cavity methods [4, 5]. The key outcome is a system of N nonlinear equations determining the chemical potentials, from which both the expected network structure and the average ground-state energy can be computed. In the zero-temperature limit, the framework yields an explicit expression for the minimum energy of a given instance, while averaging over disorder recovers known asymptotic results. Importantly, the method provides a polynomial-time approximation algorithm (scaling as O(N2)) for estimating optimal costs in random instances. For the matching and k-factor prob- lems, the approach converges to the exact optimal solution under mild conditions. For the TSP, the method becomes asymptotically exact in the thermodynamic limit and can be combined with a greedy heuristic to construct near-optimal tours.

Overall, this grand-canonical formulation provides a simple, intuitive, and analytically tractable alternative to existing methods, revealing new connections between combinatorial optimization and statistical physics.

References:

[1] Enrico Maria Fenoaltea, Izat B Baybusinov, Jianyang Zhao, Lei Zhou, and Yi-Cheng Zhang. The stable marriage problem: An interdisciplinary review from the physicist's perspective. Physics Reports, 917:1–79, 2021.

[2] Michael Held and Richard M Karp. The traveling-salesman problem and minimum spanning trees. Operations research, 18(6):1138–1162, 1970.

[3] George S Lueker. Optimization problems on graphs with independent random edge weights. SIAM Journal on Computing, 10(2):338–351, 1981.

[4] Marc M ezard and Giorgio Parisi. Replicas and optimization. Journal de Physique Lettres, 46(17):771–778, 1985.

[5] Marc Mezard and Giorgio Parisi. Mean-field equations for the matching and the travelling salesman problems. Europhysics Letters, 2(12):913, 1986.