# Exactly solvable diffusions from space-time transformations

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One of the most fundamental tools in modern non-equilibrium physics is the Langevin equation, which, mathematically speaking, is a stochastic differential equation (SDE). In this work [1], we address the problem of solving SDEs and their related first-passage time problem via a space-time transformation.

To be more detailed, we considered the most general overdamped diffusion model described by the Ito SDE $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$, where $W_t$ is the standard Wiener process, while $\mu$ and $\sigma$ are generic functions of position and time. We obtain a specific condition that $\mu$ and $\sigma$ must fulfil in order to be able to solve the SDE via mapping the generic process, using a suitable space-time transformation, onto the simpler Wiener process. By taking advantage of this transformation, we obtain the propagator in the case of open, reflecting, and absorbing time-dependent boundary conditions for a large class of diffusion processes.

In particular, this allows us to derive the first-passage time statistics of such a large class of models, some of which were so far unknown. While our results are valid for a wide range of non-autonomous, non-linear and non-homogeneous processes, we illustrate applications in stochastic thermodynamics by focusing on the propagator and the first-passage-time statistics of isoentropic processes that were previously realised in the laboratory by Brownian particles trapped with optical tweezers. For the aforementioned process, we were also able to derive analytically the distribution of the stochastic heat production.

The results of this work have been applied in [2] to derive analytical expressions for the power and efficiency in a colloidal heat engine driven by a feedback protocol.

References:

[1] Di Bello C, Roldán É and Metzler R 2025 New J. Phys. 27 075604.

[2] Tohme T, Bedoya V, Di Bello C, Bresque L, Manzano G and Roldán É 2025 Phys. Rev. Lett. 135, 067101.