Bayesian Field Theory of the Rate Estimation

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Reconstruction of time-dependent signals from discrete, noisy data is a fundamental challenge across numerous scientific disciplines, including neuroscience, cosmology, and bacterial chemotaxis. In these inverse problems, researchers must infer an unobserved signal based on a prior model and a measurement model of the data likelihood conditional on the signal. To address the statistical inference of a time-dependent rate of events, this paper leverages the robust framework of Bayesian Field Theory (BFT). BFT provides a principled methodology for statistical inference by treating the posterior as a Boltzmann distribution and employing field theory techniques, such as perturbative expansions, to achieve approximations beyond the standard linear regime while gaining deep physical insight.

In this work, the underlying dynamics of the time-dependent rate are modeled using a Geometric Brownian Motion (GBM), while the observed events are assumed to be stochastically generated by the corresponding inhomogeneous Poisson process. For a fixed, known volatility, the posterior rate path distribution, conditional on the observed events realization, is exactly mapped to equilibrium Langevin field fluctuations within a potential. Consequently, the specific configuration that minimizes this potential corresponds to the maximum likelihood (ML) solution for the log-process, while the fluctuations around it encode the target distribution.

Going beyond the local linear reference regime, the resulting Langevin equation involves nonlinearities and an explicit dependence on the local shape of the maximum likelihood curve. We systematically study these impacts by developing a perturbative expansion and formulating a specific scaling hypothesis to account for the order of shape corrections. Through this analytical derivation, we uncover several key findings. First, we demonstrate that pure nonlinearities, which arise from the non-harmonic local potential, heavily dominate the corrections to the mean and skewness. Crucially, we also reveal that the leading shape correction to the variance is directly driven by noise propagation originating from the signal's effective curvature. This shape correction significantly affects the variance by propagating noise from neighboring regions, effectively expanding the error tube at the peaks of the maximum likelihood curve.

To validate these theoretical derivations, we conduct extensive numerical simulations, successfully verifying the derived expansion. Furthermore, we effectively illustrate the practical significance and real-world applicability of our proposed Bayesian Field Theory approach using high-precision neural spike recordings obtained from the Allen Brain Observatory. Overall, our work provides a profound analytical understanding of how non-harmonic potentials and maximum likelihood curve geometries actively influence the moments of the posterior distribution, offering a more precise, physics-inspired framework for signal reconstruction in complex, time-dependent rate estimation challenges.