Second order structure of a finite sample space exponential manifold

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On a finite sample space Ω, all strictly positive densities have the exponential form $q = \exp(u - K_p(u)) = e_p(u)$, where p is a strictly positive reference density, u is a random variable such that $E_p(u) = 0$, $K_p(u)$ is a normalising constant. As $u = \log(q/p) - E_p(\log(q/p))$, all densities are parameterized by the chart $s_p\colon q \mapsto u$. Given a subspace $V_p \subset L_0^2(p)$ of p-centered random variables, the exponential family $\mathcal{E}$ is the family of densities $e_p(v)$, $v \in V$. In the atlas of the charts s_q, $q \in \mathcal{E}_p(V)$, the transition mappings from the reference p to the reference q are $V \ni u \mapsto \exp(u - K_p(u)).p \mapsto u - K_p(u) - \log(q/p) = u - E_q(u) - (E_q(u) + K_p(u))$. Hence, we have an *affine atlas*, with *exponential transport* $U_p^q\colon V_p \ni u \mapsto u - E_q(u) \in V_q$. The velocity of a curve $t \mapsto p(t)$ in the chart centered at p is $(d/dt)(log(p(t)/p) - E_p(\log(p(t)/p))) = \dot{p}(t)/p(t) - E_p(\dot{p}(t)/p(t)) = \dot{u}(t) - E_{p(t)}(\dot{u}(t)) = U_p^{p(t)}\dot{u}(t)$. The velocity at $t = 0$ in the chart $s_{p(0)}$ is $Dp(0) = \dot{u}(0)$ so that the expression of the *tangent space* at p is $T_p\mathcal{E} = V_p$. The expression of the tangent bundle is the *statistical bundle* $S\mathcal{E}$ consisting of of couples (q, v), $q \in \mathcal{E}$ and $v \in V_q$. The *statistical gradient* of a mapping $\phi\colon \mathcal{E} \to \mathbb{R}$ is the section $\operatorname{grad}\phi$ such that $(d/dt)\phi[p(t)] = E_{p(t)}(\operatorname{grad}\phi(p(t))(D/dt)p(t))$. There is a Riemnnian metric $V_p \times V_p \ni (u, v) \mapsto E_p(uv)$ and the dual of the exponential transport is the *mixture transport.* The Levi-Civita connection and the two affine transports define three different geometries on the statistical bundle. The different covariant derivatives and their Hessians result. We consider applications of this set-up and shortly discuss the relation of these geometries with the Gini dissimilarity index.

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