The gradient-flow equations in information geometry: some approaches from geometrical optics to Randers-Finsler geometry

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Information geometry (IG) is a useful and powerful framework for studying some families of probability distributions by identifying the space of probability distributions with a differentiable manifold endowed with a Riemannian metric and an affine connection which is not necessarily Levi-Civita connection. In IG, a Riemann metric is obtained from the Hessian of a potential function, and the fluctuations play important role. Especially so-called fluctuation-response relations are related with the Hessian metric.

On the other hand it is known that the gradient-flow equations are useful for some optimization problems. The gradient flows on a Riemann manifold follow the direction of gradient descent (or ascent) in the landscape of a potential functional, with respect to the curved structure of the underlying metric space. The information geometric studies on the gradient systems were originally done by independently Nakamura, and Fujiwara and Amari. A remarkable feature of their works is that a certain kind of gradient flow on a dually flat space can be expressed as a Hamilton flow. Later, several works on this issue have been done from the different perspectives. Boumuki and Noda studied the relationship between Hamiltonian flow and gradient flow from the perspective of symplectic geometries. The same issue was studied in from the perspective of geometric optics. The gradient-flows in IG are related to the thermodynamic processes. The analytical mechanical properties concerning the gradient-flow equations in IG are studied and discussed the deformations of the gradient-flow equations which lead to Randers-Finsler metrics.

In this contribution, we first explain some basics of the gradient-flow equations in IG. Then we show the geometrical optics approach based on the generalized eikonal equation, which is equivalent to Hamilton-Jacobi equation. Here null (or light-like) geodesics play central role. It is shown that the gradient-flows in IG are related to the Hamilton-flows associated with the so-called geodesic Hamiltonians. In conventional way of IG, the natural coordinate (θ- and η-) spaces are characterized with α-connections, which provide the parallel translation rule in these spaces. In addition, unlike Riemann geometry, the Hessian metrics g are used to determine the orthogonality only. The θ- and η-coordinate systems are regarded as the different coordinates on the same manifold. In contrast, in our perspective, the θ- and η-coordinate systems are regarded as the different spaces or manifolds. We will show some examples, e.g., the gradient-flows for the normal (Gaussian) probability distribution functions.

References

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