Approximation of functional differential equations

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The fundamental importance of functional differential equations (FDEs) has been recognized in many areas of mathematical physics. A classical example is the Hopf characteristic functional equation of turbulence (Hopf, 1952). Such functional equation encodes the full statistical information of the stochastic solution to the Navier-Stokes equation, including multi-point statistical moments, cumulants, and multi-point joint probability density functions. For this reason, the Hopf characteristic functional equation was deemed by Monin and Yaglom (1972) to be ``the most compact formulation of the general turbulence problem'', which is the problem of determining the statistical properties of the velocity and the pressure fields solving the Navier-Stokes equations, given statistical information on the initial state. Another well-known example of functional differential equation is the Schwinger-Dyson equation of quantum field theory. Such equation describes the dynamics of the generating functional of the Green functions of a quantum field theory, allowing us to propagate quantum field interactions in a perturbation setting (e.g., with Feynman diagrams), or in a strong coupling regime. FDEs have recently appeared also in mean-field games, and mean-field optimal control. Mean-field games are optimization problems involving a large (potentially infinite) number of interacting players. In some cases, it is possible to reformulate such optimization problems in terms of a nonlinear Hamilton-Jacobi-Bellman FDE in a Wasserstein space of probability measures. This opens the possibility to formulate, e.g., deep learning as a mean-field optimal control problem via a functional HJB equation (E, Han & Li 2018).

Computing the solution to FDEs such as the Hopf characteristic functional equation or the functional HJB equation is a long-standing problem in mathematical physics. Back in 1972 Monin and Yaglom stated that: ``When we tried to develop a complete statistical description of turbulence with the aid of the Hopf equation for the characteristic functional we found that no general mathematical formalism for solving linear equations in functional derivatives was available.'' Since then there were of course advances in the theory of existence and uniqueness of solution to both linear and nonlinear FDEs, and also their approximation.

In this talk, I will present recent results on approximation theory of nonlinear functionals, functional derivatives, and functional differential equations (FDEs) in Hilbert and Banach spaces. The purpose of this analysis is twofold: first, we prove that continuous nonlinear functionals, functional derivatives and FDEs can be approximated uniformly on any compact subset of a real Banach space admitting a basis by high-dimensional multivariate functions and high-dimensional partial differential equations (PDEs), respectively. Second, we show that the convergence rate of such functional approximations can be exponential, depending on the regularity of the functional (in particular its Frechet differentiability), and its domain. We also provide necessary and sufficient conditions for consistency, stability and convergence of approximations to linear FDEs. These results open the possibility to utilize numerical techniques for high-dimensional systems such as deep neural networks and numerical tensor methods to approximate nonlinear functionals in terms of high-dimensional functions, and compute approximate solutions of FDEs by solving high-dimensional PDEs.