From financial risk measures to thermodynamic formalism

Miguel Mendes, Maria Carvalho, Paulo Varandas, Andrzej Bis
Universidade Do Porto, Porto, Portugal

In Artzner et al. [1] an axiomatic approach to measure the risk of financial portfolios was introduced. Namely, a functional acting on the space of bounded random variables is said to be a coherent risk measure if it is invariant under translations, positively homogeneous, subadditive and monotone. Interpretations of these abstract properties in the realm of Finance were conveniently presented but the main purpose of the axiomatic setting was to obtain a robust representation theorem which establishes a one-to-one correspondence between coherent risk measures and finitely additive probabilities. More precisely, the risk of a financial position X is the maximum expected value of -X attained on a specific convex set of finitely additive probability measures.

Followingly, Follmer and Scheid [2], generalized the previous result proving a robust representation theorem for convex, translation invariant and positively homogeneous functionals. In this case the risk of X is measured by the expected values of -X “penalised” by a convex and lower semicontinuous functional acting on the space finitely additive probability measures. This functional turns out to be the Fenchel-Legendre transform of the risk measure.

This result is of great generality and provides a variational principle ready for use in the context of thermodynamic formalism in dynamical systems. Immediately, one notes that the variational principle is valid for generalised pressure functions, i.e., convex, translation invariant and positively homogeneous functionals. Differentiability properties of these pressure functions are obtained analogously to the ones proved in Walters [3]. Then we can apply the robust representation theorem given by Follmer and Schied to the topological pressure acting on the space of continuous potentials. As it is well known (see, e.g., Walters [4] p.214) the topological pressure satisfies the three properties of generalised pressure functions. We then obtain a variational principle where the Kolgomorov-Sinai metric entropy is replaced by the Fenchel-Legendre conjugate of the topological pressure. Please see Ref. [5] for more details.

References

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